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Path-integral forms for the Klein–Gordon wavefunction†

Frank G Krausz
Lyman Laboratory of Physics, Harvard University, Cambridge, MA 02138, USA

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Abstract. We discuss how to define the path integral $\int \mathscr{D}x\, \mathrm{e}^{\mathrm{i}S}$ when S is the Klein–Gordon action and show that it is possible to use this form as a Euclidean-space wavefunction.

1. Introduction

There are two path-integral forms for the Klein–Gordon wavefunction: the intuitively obvious

$$K(x)=\int \mathscr{D}x\, \mathrm{e}^{\mathrm{i}S}=\int \mathscr{D}x \exp\left\{\mathrm{i}\int \mathrm{d}u\left[m\left(\frac{\mathrm{d}x^\mu}{\mathrm{d}u}\frac{\mathrm{d}x^\mu}{\mathrm{d}u}\right)^{1/2}+eA_\mu\frac{\mathrm{d}x^\mu}{\mathrm{d}u}\right]\right\} \tag{1}$$

and Feynman's form (Feynman 1950),

$$\Psi(x)=\int_0^\infty \mathrm{d}u_0 \exp(-\tfrac{1}{2}m^2u_0)\int \mathscr{D}x \exp\left\{-\mathrm{i}\int_0^{u_0}\mathrm{d}u\left[\frac{1}{2}\left(\frac{\mathrm{d}x^\mu}{\mathrm{d}u}\right)^2+eA_\mu\frac{\mathrm{d}x^\mu}{\mathrm{d}u}\right]\right\}. \tag{2}$$

These are known to be formally equivalent, by an application of the Fade'ev–Popov trick: one observes the exponent of $K(x)$ to be reparametrisation invariant and, roughly speaking, divides out the volume of the reparametrisation group (Bardacki and Samuel 1978).

But $K(x)$ is not defined *a priori*, not being of the usual Gaussian type. We discuss the defining of this integral and show that a sensible choice of definitions preserves the equality with $\Psi(x)$, at least in Euclidean space. We show that neither expression is useful in Minkowski space.

2. Defining $K(x)$

The natural reparametrisation invariance of relativistic actions alluded to above will prevent us from following the usual procedure in defining our path integral. If u is an arbitrary parameter, we obtain, for $A_\mu=0$,

$$\begin{aligned}K(x_N-x_0)&=\lim_{N\to\infty}\int\prod_{i=1}^{N-1}\mathrm{d}x_i^\mu(u)\exp\left[\mathrm{i}m\varepsilon\left(\frac{\sum_{i=1}^{N}|x_i-x_{i-1}|}{\varepsilon}\right)\right](\mathcal{N}(\varepsilon))^{N-1}\\&=\lim_{N\to\infty}\int\prod_{i=1}^{N-1}\mathrm{d}x_i^\mu(u)\exp\left(\mathrm{i}m\sum_{i=1}|x_i-x_{i-1}|\right)(\mathcal{N}(\varepsilon))^{N-1}\end{aligned} \tag{3}$$

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where $\varepsilon = (\text{range of } u)/N$ and $\mathcal{N}(\varepsilon)$ is a normalising factor. But since we have a reparametrisation-invariant action, ε has disappeared from the exponent. Whatever procedure we try to use to make the exponential damp for large values of $|x_i - x_{i-1}|$, there is nothing to set the scale of the damping, so no $\mathcal{N}(\varepsilon)$ can possibly be chosen to make the integral finite.

The obvious cure for this disease is to enforce a specific parametrisation: for example, if we parametrise by the time coordinate, expression (3) becomes (excluding tachyonic paths and letting $\boldsymbol{\eta}_i \equiv \boldsymbol{x}_i - \boldsymbol{x}_{i-1}$, $\boldsymbol{x} \equiv \boldsymbol{x}_N - \boldsymbol{x}_0$)

$$K = \lim_{N\to\infty} \int_{|\boldsymbol{\eta}_i| \le \varepsilon} \prod_{i=1}^{N} \mathrm{d}^3\boldsymbol{\eta}_i \, \delta^3\left(\sum_{i=1}^{N} \boldsymbol{\eta}_i - \boldsymbol{x}\right) \exp[\mathrm{i}m(\varepsilon^2 - \eta_i^2)^{1/2}](\mathcal{N}(\varepsilon))^{N-1}. \tag{4}$$

But attempts based on the use of time as a parameter are doomed to fail. Following Feynman's procedure for the Schrödinger equation (Feynman 1948), if $x_N^0 - x_0^0 \equiv t$,

$$K(t+\delta t, \boldsymbol{x}) = \int_{|\boldsymbol{\eta}| \le \delta t} K(t, \boldsymbol{x} - \boldsymbol{\eta}) \exp[\mathrm{i}(\delta t^2 - \eta^2)^{1/2}] \, \mathrm{d}^3\boldsymbol{\eta} \mathcal{N}(\delta t). \tag{5}$$

Expanding both sides in powers of δt gives a PDE with a $\partial K/\partial t$ term—certainly not the Klein–Gordon equation. Of course, had we been so foolish as to do either the non-relativistic or the relativistic case with, say, the z coordinate as a parameter, we would have obtained, in a similar way, a $\partial K/\partial z$ term. The reason for the impropriety of such a parametrisation is that one does not in general expect the other coordinates to be single-valued functions of z. That a t parametrisation fails to give the Klein–Gordon equation implies that, if (3) is to hold for arbitrary parametrisation, $\boldsymbol{x}(t)$ is ill defined in the relativistic case: one *must* sum over paths going back and forth in time. (The presence of the $\partial/\partial t$ term in the Dirac equation is perhaps related to the fact that for fermions, particles and antiparticles are necessarily distinct, so simply treating them together in this way is wrong.)

A better choice of parameters is the proper time, s. But Feynman's prescription for a sum-over-histories calls for a *fixed* parametric interval (in the case of the Schrödinger equation, the time interval), which would constrain us to summing only over paths of a fixed four-length l. Since we want to include paths of different four-lengths in our sum, we will have to integrate separately over l. This will have the advantage of eliminating the $\mathrm{d}/\mathrm{d}s$ term from the PDE. A difficulty with this choice is that, unless one excludes paths with light-like segments (which, because of the *zitterbewegung*, seems wrong) it would appear that this parametrisation too is ill defined, except in Euclidean space–time.

But if we look at equation (3) in this parametrisation (for the case of $A = 0$) we obtain

$$K = \lim_{N\to\infty} \int_0^L \mathrm{d}l \, \mathrm{e}^{\mathrm{i}ml} \int \prod_{i=1}^{N} \mathrm{d}^4\eta_i \, \delta\left(|\eta_i| - \frac{l}{N}\right) \delta^{(4)}\left(\sum_{i=1}^{N} \eta_i - x\right) \left[\mathcal{N}\left(\frac{l}{N}\right)\right]^{N-1}. \tag{6}$$

Restricting $|\eta_i| = l/N$ as we do here, or damping $|\eta_i|$ in some other version of equation (3), places no constraint on $\int \mathrm{d}^3\boldsymbol{\eta}_i$ if we are in Minkowski space, so, again, no $\mathcal{N}(\varepsilon)$ can make the integral finite. It is therefore always necessary to let $|\eta_i|$ be the Euclidean length (and set $L = \infty$).

3. Deriving Feynman's form

Let us proceed to derive equation (2), taking as our starting point

$$K=\int_0^\infty \mathrm{d}l \int \mathscr{D}x \exp\left[\mathrm{i}\int \left(m\left|\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\right|_{\mathrm{E}}+eA_\mu\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\right)\mathrm{d}s\right]\text{‘}\delta\left(\int\left|\frac{\mathrm{d}x}{\mathrm{d}s'}\right|_{\mathrm{E}}-s\right)\text{’} \tag{7}$$

where the expression

$$\text{‘}\delta\left|\int\left|\frac{\mathrm{d}x}{\mathrm{d}s'}\right|_{\mathrm{E}}-s\right|\text{’}$$

represents our choice of parametrisation and $|y|_{\mathrm{E}}$ = the Euclidean length of y. Using the fact that $|\mathrm{d}x^\mu/\mathrm{d}s|_{\mathrm{E}}\equiv 1$ we write, somewhat arbitrarily,

$$K=\int_0^\infty \mathrm{d}l\, \mathrm{e}^{\mathrm{i}ml/2}\int \mathscr{D}x \exp\left[\mathrm{i}\left(\int_0^l\left|\frac{m}{2}\right|\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\right|^2+eA\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\right)\mathrm{d}s\right]\text{‘}\delta\left(\int\left|\frac{\mathrm{d}x}{\mathrm{d}s'}\right|-s\right)\text{’}. \tag{8}$$

Now we use a Gaussian form for our ‘Dirac delta’:

$$\begin{aligned}K=\lim_{\substack{N\to\infty\\ \eta\to 0}}\int_0^\infty \mathrm{d}l\,\mathrm{e}^{\mathrm{i}ml/2}\int \Pi\,\mathrm{d}x_i(s)\exp\bigg\{\mathrm{i}\bigg[&\frac{m}{2}\frac{(x_i-x_{i-1})^2}{\varepsilon}\\ &+e\left(\frac{A_\mu(x_i)+A_\mu(x_{i-1})}{2}\right)(x_i-x_{i-1})\\ &+\left(\frac{(\varepsilon-|x_i-x_{i-1}|)^2}{\mathrm{i}\eta}\right)\bigg]\bigg\}\frac{1}{(\pi\eta)^{1/2}}(\mathcal{N}(\varepsilon))^{N-1}.\end{aligned} \tag{9}$$

Define $2\varepsilon/\eta=\xi$, and take limits so that $\xi\to 0$.

$$\begin{aligned}K=\lim\int_0^\infty \mathrm{d}l\,\mathrm{e}^{\mathrm{i}ml/2}\int \Pi\,\mathrm{d}^4x_i(s)\exp\mathrm{i}\bigg\{&\frac{m-\mathrm{i}\xi}{2}\frac{(x_i^\mu-x_{i-1}^\mu)^2}{\varepsilon}\\ &+e\frac{[A_\mu(x_i)+A_\mu(x_{i-1})]}{2}(x_i^\mu-x_{i-1}^\mu)\\ &-\mathrm{i}\xi|(x_i-x_{i-1})|+\mathrm{i}\varepsilon\xi\bigg)\bigg\}\frac{1}{(\pi\eta)^{1/2}}(\mathcal{N}(\varepsilon))^{N-1}\end{aligned} \tag{10a}$$

$$\begin{aligned}=\lim_{\varepsilon,\xi\to 0}\int_0^\infty \mathrm{d}l\,\mathrm{e}^{\mathrm{i}(m-2\xi)l/2}\int \Pi\,\mathrm{d}x_i(s)\tilde{\mathcal{N}}(\varepsilon)^{N-1}\\ \times\exp\left\{\mathrm{i}\left[\frac{m-\mathrm{i}\xi}{2}\frac{(x_i-x_{i-1})^2}{\varepsilon}+e\left(\frac{A_\mu(x_i)+A_\mu(x_{i-1})}{2}\right)(x_i-x_{i-1})\right]\right\}\end{aligned} \tag{10b}$$

where

$$\tilde{\mathcal{N}}(\varepsilon)=(\varepsilon)/(\pi\eta)^{1/2},$$

which we may do since, having already insisted that $\varepsilon/\eta\to 0$, we may with no loss of generality take, say, $\eta=\varepsilon^{1/2}$. So, finally,

$$K=\lim_{\xi\to 0}\int_0^\infty \mathrm{d}l\,\mathrm{e}^{\mathrm{i}(m-2\xi)l/2}\int \mathscr{D}x\exp\left\{\mathrm{i}\int_0^l \mathrm{d}s\left[\frac{m-\mathrm{i}\xi}{2}\left(\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\right)^2+eA_\mu\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\right]\right\}. \tag{11}$$

We can make this resemble equation (2), as promised, by a change of variables. Let $u_0 = l/m$, $u = s/m$, and absorb an extra $1/m$ in the measure $\mathscr{D}x$:

$$K = \lim_{\xi\to 0} \int_0^\infty \mathrm{d}u_0 \exp[\mathrm{i}(m^2 - 2\xi m)u_0/2]$$
$$\times \int \mathscr{D}x \exp\left\{\mathrm{i}\int_0^{u_0} \left[\left(\frac{1-\mathrm{i}\xi/m}{2}\right)\left(\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\right)^2 + eA_\mu \frac{\mathrm{d}x^\mu}{\mathrm{d}s}\right]\right\}. \tag{12}$$

We will see below that this differs from equation (2) mainly in an inessential sign convention.

4. Feynman's derivation of the Klein–Gordon equation

Define a function $\phi(x^\mu, u)$ by

$$K = \lim_{\eta\to 0} \int_0^\infty \mathrm{d}u \exp[\mathrm{i}(m^2 - \mathrm{i}\eta)u/2]\phi(x, u). \tag{13}$$

If we look at the path integral form of ϕ and neglect the $\mathrm{i}\xi/2m$ term, we see that it must satisfy a Schrödinger-like equation with u as the time:

$$\mathrm{i}\,\partial\phi/\partial u = \tfrac{1}{2}(\mathrm{i}\,\partial/\partial x_\mu - A_\mu)^2\phi. \tag{14}$$

Feynman's starting point is to *define* ϕ to satisfy (14) (but with a relative minus sign between the RHS and LHS). It is amusing to notice that the usual derivation of (14) fails in Minkowski space. One has that

$$\phi(x^\mu, u+\delta u) = \int \phi(x^\mu - \delta x^\mu, u) \exp\left\{\mathrm{i}\delta u\left[\left(\frac{\delta x^\mu}{\delta u}\right)^2 + eA_\mu \frac{\delta x^\nu}{\delta u}\right]\right\} \mathrm{d}^4(\delta x^\mu).$$

The next step is to expand both sides in δu and compare coefficients, using the Gaussian damping to obtain $\delta x^\mu \sim (\delta u)^{1/2}$. But in fact the damping only forces $(\delta x^\mu)^2 \sim \delta u$, which in Minkowski space does not imply what we need.

Feynman then asserts that we project out the eigenfunctions $\phi(m^2, x, u)$ satisfying

$$\partial\phi(m^2)/\partial u = -\tfrac{1}{2}\mathrm{i}m^2\phi(m^2) \tag{15}$$

by writing

$$\exp(\tfrac{1}{2}\mathrm{i}m^2 u)\phi(m^2) = K = \int_0^\infty \exp(\tfrac{1}{2}\mathrm{i}m^2 u)\phi(x, u)\,\mathrm{d}u. \tag{16}$$

(Feynman's convention is to project out $\phi(-m^2)$.) Equations (16) and (14) (with either sign convention) imply the Klein–Gordon equation. Note that it does us no good to try to take a *Fourier* transform of ϕ since $\phi(x, u) \equiv 0$ for $u < 0$. The $\mathrm{i}\eta$ in equation (13) now appears as the usual real part of the exponent in a Laplace transform.

5. Discussion

It has been emphasised (notably by Wu and Yang (1975)) that in gauge theories the quanitity $\phi = P \exp(\mathrm{i}e \int \hat{A}_\mu\, \mathrm{d}x_\mu)$ is the physically relevant one. We would like to point out that our construction legitimises the idea that in general $\exp(\mathrm{i}S)$ is the 'amplitude for a path' for Klein–Gordon particles, and suggests that vector potentials, for which S is reparametrisation invariant, are the natural ones. It is remarkable to us how the statement that the wavefunction is $\int \mathscr{D}x\ \mathrm{e}^{\mathrm{i}S}$ survives redefinition, and we wonder if an extension to the Dirac equation might not after all exist. (The free Dirac particle in two dimensions is discussed in Feynman and Hibbs (1965).)

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